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Technical Notes

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Low-Reynolds-Number k–$\tilde{\epsilon}$ Model with Enhanced Near-Wall Dissipation

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Nomenclature

C_f	=	friction coefficient
C_μ	=	eddy-viscosity coefficient
E	=	source term in dissipation equation
f_ϵ, f_μ	=	viscous damping functions
h	=	step height
k	=	turbulent kinetic energy
Nu	=	local Nusselt number
P	=	turbulent production term
S	=	mean strain-rate invariant
T_t	=	realizable timescale
t	=	time
U_i	=	mean velocity components
$-\overline{u_i u_j}$	=	Reynolds stresses
W	=	mean vorticity invariant
x_i	=	Cartesian coordinates
y^+	=	nondimensional normal distance from wall
ϵ	=	turbulent dissipation
μ, μ_T	=	laminar and eddy viscosities
ν	=	molecular kinematic viscosity
ρ	=	density
σ	=	turbulent Prandtl number

Introduction

THE modeling of near-wall turbulence usually involves distance to the wall as an explicit parameter. This often renders the model inappropriate to simulate complex flows involving multiple surfaces, the wall distance of which becomes cumbersome to define. A remedy for this flaw is to develop a model that implicates no explicit wall distance while integrating it toward the solid surface.

A wall-distance-free, low-Reynolds-number k–$\tilde{\epsilon}$ turbulence closure model is developed wherein an $\tilde{\epsilon} = \epsilon - 2\nu(\partial\sqrt{k}/\partial x_j)^2$ equation is adopted instead of ϵ equation. It incorporates an extra source term in the $\tilde{\epsilon}$ transport equation that augments the dissipation level in nonequilibrium flow regions, thus reducing the turbulent kinetic energy and length scale magnitudes to improve prediction of adverse pressure gradient flows involving separation and reattachment. The wall singularity is removed by using a physically appropriate timescale that never falls below the Kolmogorov (dissipative eddy) timescale, representing timescale realizability enforcement accompanied by the near-wall turbulent phenomena. An eddy-viscosity damping function is designed in terms of total kinetic energy and the invariants of strain rate and vorticity tensors with no reference to the distance from the wall. In addition, the turbulent Prandtl number σ_k is adjusted to provide substantial turbulent diffusion in the near-wall region. In essence, the model is tensorially invariant, frame indifferent, and applicable to arbitrary topologies. The performance of the new model is demonstrated through the comparison with experimental and direct numerical simulation (DNS) data of well-documented flows.

Present Model

The proposed model determines the turbulence kinetic energy k and its dissipation rate $\tilde{\epsilon}$ by the following transport equations:

$$\frac{\partial \rho k}{\partial t} + \frac{\partial \rho U_j k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_T}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + \rho P - \rho\epsilon \tag{1}$$

$$\frac{\partial \rho \tilde{\epsilon}}{\partial t} + \frac{\partial \rho U_j \tilde{\epsilon}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_T}{\sigma_\epsilon}\right)\frac{\partial \tilde{\epsilon}}{\partial x_j}\right] + \frac{C_{\epsilon 1}\rho P - C_{\epsilon 2}\rho\tilde{\epsilon} - f_\epsilon \rho D + E}{T_t} \tag{2}$$

where $\epsilon = \tilde{\epsilon} + D$ and the turbulent production term $P = -\overline{u_i u_j}\,(\partial U_i/\partial x_j)$. The eddy viscosity and other variables are evaluated as

$$\mu_T = C_\mu f_\mu \rho k T_t, \qquad T_t = \max\left(k/\tilde{\epsilon},\, C_T\sqrt{\nu/\epsilon}\right)$$

$$D = 2\nu\left(\partial\sqrt{k}\big/\partial x_j\right)^2, \qquad f_\epsilon = 1.0 - \sqrt{f_k} \tag{3}$$

The function f_k is defined later. The realizable timescale T_t prevents the singularity at $y_n = 0$ in the dissipation equation, where y_n is the normal distance from the wall. The associated constants are $C_\mu = 0.09$, $C_{\epsilon 1} = 1.45$, $C_{\epsilon 2} = 1.9$, $\sigma_\epsilon = 1.4$, and $C_T = \sqrt{2}$.

As a pragmatic device, the damping functions are chosen to be a function of R_λ, given by

$$f_\mu = \tanh\left(3.0 \times 10^{-3} R_\lambda + 5.5 \times 10^{-4} R_\lambda^2\right)$$

$$R_\lambda = \sqrt{R_k/(1 + T_t\eta)}$$

$$\sigma_k = 0.2 + 1.3 f_k^2, \qquad f_k = 2\sqrt{f_\mu}\big/(1 + f_\mu) \tag{4}$$

where $R_k = T_t K_T/\nu$ and $K_T = \boldsymbol{U}\cdot\boldsymbol{U}/2 + k$. The parameter $\eta = \max(S, W)$ contains the invariants $S = \sqrt{(2S_{ij}S_{ij})}$ and $W = \sqrt{(2W_{ij}W_{ij})}$. The mean strain rate and mean vorticity tensors S_{ij} and W_{ij} are defined as

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \qquad W_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right) \tag{5}$$

The empirical function f_μ is valid in the whole flowfield, including the viscous sublayer and the logarithmic layer. In the region close to the wall, the Reynolds stress $-\overline{uv} \sim y^3$ and $k \sim y^2$. To preserve the correct cubic power law behavior of $-\overline{uv}$, the damping function needs to increase proportionally to y in the near-wall region. Equation (4) confirms that, as $y \to 0$, $R_\lambda \sim y$ and, hence, $f_\mu \sim y$. Alternatively, the adopted form of f_μ reproduces correctly the asymptotic limit, involving the distinct effects of low Reynolds

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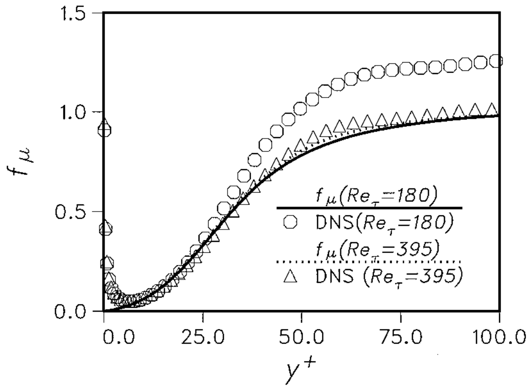

Fig. 1 Variation of damping function with wall distance in channel flow.

number and wall proximity. As evinced by Fig. 1 in comparison with the DNS data[1] for fully developed turbulent channel flows, the proposed function approaches 1 far from the wall to ensure the model being compatible to the standard k–ϵ turbulence model. The use of R_λ dose not meet the singularity at the separating or the reattaching point, in contrast to the adoption of $y^+ = u_\tau y/\nu$, where u_τ is the friction velocity. Consequently, the model is applicable to separated and reattaching flows. In principle, the eddy viscosity envisages two separate effects, comprising the influences of low Reynolds number and wall proximity. To this end, note that the asymptotic value of turbulent Prandtl number σ_k is adopted as 0.2 to obtain sufficient diffusion in the vicinity of the wall.[2]

The extra source term E in Eq. (2) is constructed to account for optimal results compared with DNS and experimental data:

$$E = \max(0, E_k/C_k, C_\epsilon E_\epsilon), \qquad C_k = C_\epsilon/2.0, \qquad C_\epsilon = 4.5 \tag{6}$$

where the quantities E_k and E_ϵ originate from the most extensive turbulent diffusion models for k and ϵ equations derived by Yoshizawa[3] with the two-scale direct-interaction approach using the inertial-range simplification. To receive positive benefits from the numerical reliability and to integrate the inertial-range condition directly to the solid wall, the cross-diffusion terms are designed as[4]

$$E_k = -\mu_T\left[\frac{\partial(k/\epsilon)}{\partial x_j}\frac{\partial\tilde{\epsilon}}{\partial x_j}\right], \qquad E_\epsilon = \frac{\mu_T}{T_t}\left[\frac{\partial(k/\epsilon)}{\partial x_j}\frac{\partial k}{\partial x_j}\right] \tag{7}$$

The constants associated with Eq. (6) are tuned to match well-known flows considered hereinafter. Obviously, the source term E stimulates the energy dissipation in nonequilibrium flows, thereby reducing the departure of the turbulent length scale from its local equilibrium value and enabling improved prediction of adverse pressure gradient flows.

Computations

To ascertain the efficacy of the proposed model, a few applications to two-dimensional turbulent flows consisting of a fully developed channel flow, a backward-facing step flow, and heat transfer from a circular cylinder in crossflow are considered. For comparison, calculations from the original Chien (OCH) model[5] and the modified Chien (MCH) model (see Ref. 4) are included. A cell-centered finite volume scheme combined with an artificial compressibility approach[6] is employed to solve the flow equations.

Channel Flow

The computation is carried out for a fully developed turbulent channel flow at $Re_\tau = 180$ for which turbulence quantities are attainable from the DNS data.[1] Calculations are conducted in the half-width of the channel, imposing cyclic boundary conditions except for the pressure. The length of the computational domain is 32δ, where δ is the channel half-width. A 96×64 nonuniform grid refinement is considered based on the grid-independence test. To ensure the resolution of viscous sublayer, the first grid node near the wall is placed at $y^+ \approx 0.3$. Comparisons are made by plotting the results in wall units. The results shown in Fig. 2 indicate that the present

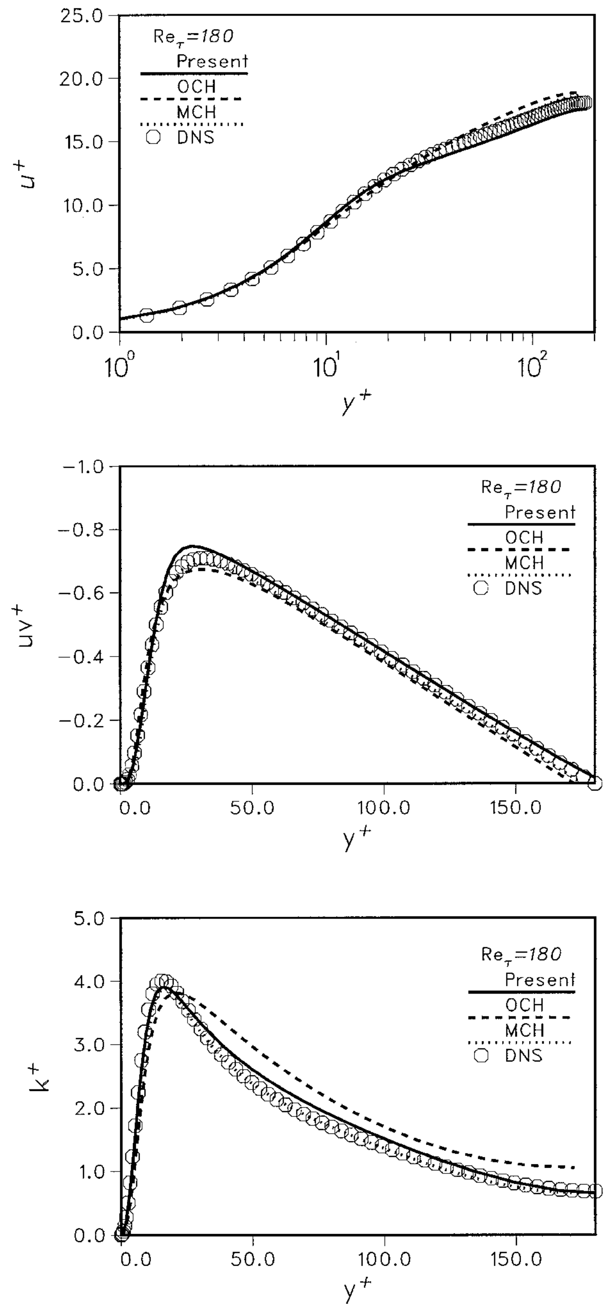

Fig. 2 Channel flow predictions compared with DNS results.

model predictions are in broad agreement with the OCH and MCH models and DNS data.

Backward Facing Step Flow

To validate the performance in complex separated and reattaching turbulent flows, the present model is applied to the flow over a backward-facing step. The computation is conducted corresponding to the experimental case with zero deflection of the wall opposite the step, as investigated by Driver and Seegmiller.[7] The ratio between the channel height and the step height h is 9, and the step height Reynolds number is $Re = 3.75 \times 10^4$. At the channel inlet, the Reynolds number based on the momentum thickness is $Re_\theta = 5 \times 10^3$. A 128×128 nonuniform grid is used for the computations, and the first near-wall grid node is at $y^+ < 1.5$. The inlet profiles for all dependent variables are generated by solving the models at the appropriate momentum thickness Reynolds number. The distance x/h shown subsequently is measured exactly from the step corner.

Computed and experimental friction coefficients C_f along the step side wall are plotted in Fig. 3. As observed, the OCH model gives the C_f distribution with a large overshoot, followed by a

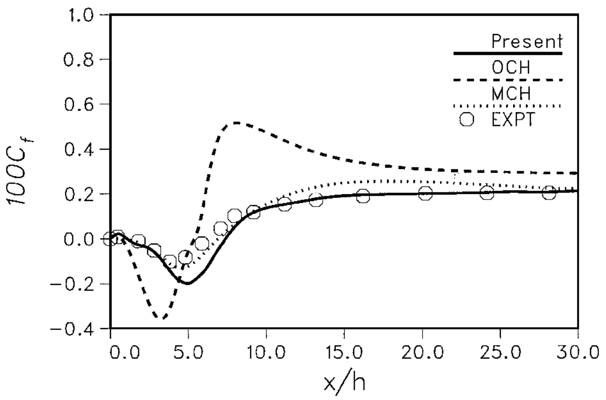

Fig. 3 Skin-friction coefficient along the bottom wall of the step flow.

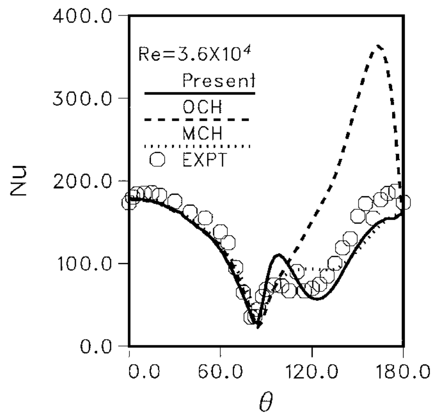

Fig. 4 Local Nusselt number distribution over half of the tube surface.

sudden drop in the immediate vicinity of the reattachment point. The positive C_f that starts from $x/h = 0$ is due to a secondary eddy that sits in the corner at the base of the step, inside the main recirculation region. The OCH model predicts a recirculation length of 5.4. The corresponding predictions by the MCH and present models are 6.8 and 6.95, respectively. The experimental value of the reattachment length is 6.26 ± 0.1, making a fairly good correspondence with the present and MCH models.

Heat Transfer from Circular Cylinder in Crossflow

The performance of the proposed model is further contrasted with the experimental data of turbulent heat transfer around a circular cylinder at $Re = 3.6 \times 10^4$ in crossflow.[8] Probably, this is a typical Reynolds number Re for practical heat exchangers. The configuration is geometrically simple but difficult to model, most likely because of the boundary-layer separation. The tested cylinder consists of a tube with $D = 0.025$ m, where D is the diameter. The reference velocity is $U_{\text{ref}} = 22.85$ m/s with an upstream turbulence intensity $Tu = 0.5\%$, defined as $Tu = \sqrt{(\frac{2}{3}k)}/U_{\text{ref}}$. An O-type grid with 128 × 96 resolution, clustered heavily near the solid wall, is employed. The radial length of the computational domain is $60D$. External boundary, that is, far-field, conditions are applied. A constant temperature is prescribed at the wall, which simulates the experimental boundary conditions. Note that the turbulent part of the total heat flux is estimated using a Boussinesq approximation and the turbulent Prandtl number $Pr_T = 0.9$.

Figure 4 shows the variation of the local Nusselt number with the azimuth angle. As can be seen, the distribution exhibits the characteristic feature of a minimum Nusselt number at the separation that corresponds to $\theta \approx 85$ deg, followed by an increase in heat transfer in the wake regions. Obviously, the present model prediction maintains good agreement with the experiment.

Conclusions

The proposed turbulent model is wall-distance free, tensorially invariant, and frame indifferent. Consequently, it is applicable to arbitrary topologies in conjunction with structured or unstructured grids. The model accounts for the near-wall and low-Reynolds-number effects emanating from the physical requirements. The potential importance of the damping functions is conspicuous. The anisotropic production in the dissipation equation is accounted for substantially by adding a secondary source term. The model is capable of evaluating the flow case including separation and reattachment.

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General Second-Order Projection Formulas for Unsteady Flows

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I. Introduction

ONE of the central issues in the design of numerical methods for the unsteady incompressible Navier–Stokes (NS) equations in primitive variables is the development of an appropriate discrete form of the incompressibility constraint if the solvability boundary conditions are given. The marker and cell (MAC) method[1] enforces the incompressibility constraint by taking the divergence of the momentum equations to derive the pressure Poisson equation. Chorin[2] developed the projection method, in which the diffusion term is implicitly updated and the time-step size is greatly enlarged. Kim and Moin (hereafter referred to as KM)[3] employed the second-order-accurate semi-implicit Crank–Nicholson scheme for the diffusion term; however, the explicit Adams–Bashforth scheme for the convective term has been shown to be weakly stable for the Navier–Stokes equation. Rai and Moin (hereafter referred to as RM)[4] employed the three-stage low-storage Runge–Kutta technique to get a high efficiency and high stability method for solving

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